Novelty Journals

ISSN 2394-7314

International Journal of Novel Research in Computer Science and Software Engineering
Vol. 9, Issue 2, pp: (37-43), Month: May - August 2022, Available at: www.noveltyjournals.com

Arbitrary Order Fractional Derivative of Inverse Fractional Trigonometric Function

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DOI: https://doi.org/10.5281/zenodo.6900497

Published Date: 25-July-2022

***Abstract*: In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative, any order fractional derivative of inverse fractional trigonometric function is obtained. The fractional binomial series and a new multiplication of fractional analytic functions play important roles in this article. In fact, these results we obtained are generalizations of those in traditional calculus. Moreover, the new multiplication is a natural operation of fractional analytic functions.**

***Keywords*: Jumarie's modified R-L fractional derivative, Inverse fractional trigonometric function, Fractional binomial series, New multiplication, Fractional analytic functions.**

I. INTRODUCTION

In applied mathematics and mathematical analysis, fractional calculus theory is used to deal with any real or complex order of derivatives or integrals. Its first appearance is in a letter written to L'Hôpital by Leibniz in 1695. Over the years, many mathematicians have used their own symbols and methods to find various definitions that conform to the concept of non integer derivative or integral. The definitions of fractional calculus mainly include Riemann-Liouville (R-L) type, Caputo type, Grunwald-Letnikov (G-L) type, Weyl type, Riesz type, Jumarie type, etc [1-5]. Fractional calculus provides a good tool to describe physical memory and heredity. Fractional calculus has been applied to many fields such as biological materials, control and robotics, viscoelastic dynamics, chaos, and quantum mechanics. Those applications have also accelerated the development of the theory of fractional calculus [6-11].

Based on Jumarie type of R-L fractional derivative, this paper obtained arbitrary order fractional derivative of inverse fractional trigonometric function. The major methods we used are the fractional binomial series and a new multiplication of fractional analytic functions. And the results obtained in this article are generalizations of the results in ordinary calculus.

II. DEFINITIONS AND PROPERTIES

The fractional calculus used in this study and some properties are introduced below.

Definition 2.1 ([12])**:** Suppose that $0<\alpha\leq 1$, and x_0 is a real number. The Jumarie's modified R-L α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt. \quad (1)$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad (2)$$

where $\Gamma(\ )$ is the gamma function. In addition, we define $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)]=\left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, and it is called the n-th order α-fractional derivative of $f(x)$, where n is any positive integer.

Proposition 2.2 ([13])**:** *Let* α, β, x_0, C *be real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

The following is the definition of fractional power series expansion of fractional function.

Definition 2.3 ([14])**:** Let x, x_0 and c_k be real numbers for all k, and $0 < \alpha \leq 1$. The series $\sum_{k=0}^{\infty} c_k (x-x_0)^{k\alpha}$ is called a real α-fractional power series. Its disk of convergence intersects the real axis in an interval $(x_0 - s, x_0 + s)$ called the interval of convergence. Each real α-fractional power series defines a real valued sum function whose value at each x in the interval of convergence is given by

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} c_k (x-x_0)^{k\alpha} \ . \tag{5}$$

This series is said to represent f_α in the interval of convergence, and it is called a α-fractional power series expansion of f_α about x_0.

Definition 2.4 ([14])**:** Let $0 < \alpha \leq 1$ and f_α be a real valued α-fractional function defined on an interval I contained in R. If f_α has α-fractional derivatives of every order at each point of I, we write $f_\alpha \in C_\alpha^\infty(I)$. If $f_\alpha \in C_\alpha^\infty(I)$ on some neighborhood of a point x_0, the series

$$\sum_{k=0}^{\infty} \frac{\left({}_{x_0}D_x^{\alpha}\right)^k [f(x)](x_0)}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \tag{6}$$

is called the α-fractional Taylor series about x_0 generated by f_α. To indicate that f_α generate this fractional Taylor series, we write

$$f_\alpha(x^\alpha) \sim \sum_{k=0}^{\infty} \frac{\left({}_{x_0}D_x^{\alpha}\right)^k [f(x)](x_0)}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{7}$$

Theorem 2.5 ([14])**:** *Suppose that* $0 < \alpha \leq 1$, *and* $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} c_k (x-x_0)^{k\alpha}$. *Then*

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{\left({}_{x_0}D_x^{\alpha}\right)^k [f(x)](x_0)}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{8}$$

Next, a new multiplication of fractional analytic functions is introduced.

Definition 2.6 ([15])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Let $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{10}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{11}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} . \end{aligned} \tag{12}$$

ISSN 2394-7314
International Journal of Novel Research in Computer Science and Software Engineering
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Definition 2.7 ([15])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha\right)^{\otimes k}, \tag{13}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x - x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha\right)^{\otimes k}. \tag{14}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}(g_\alpha(x^\alpha))^{\otimes k}, \tag{15}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}(f_\alpha(x^\alpha))^{\otimes k}. \tag{16}$$

Definition 2.8 ([15])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions at x_0 satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha. \tag{17}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

The followings are some fractional analytic functions.

Definition 2.9 ([16])**:** If $0 < \alpha \leq 1$, x is a real number, and x^α exists. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{18}$$

Remark 2.10: The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

Definition 2.11([16])**:** The α-fractional cosine and sine function are defined respectively as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k\alpha+1)} x^{2k\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2k}, \tag{19}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma((2k+1)\alpha+1)} x^{(2k+1)\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes(2k+1)}. \tag{20}$$

In addition,

$$sec_\alpha(x^\alpha) = (cos_\alpha(x^\alpha))^{\otimes -1} \tag{21}$$

is called the α-fractional secant function.

$$csc_\alpha(x^\alpha) = (sin_\alpha(x^\alpha))^{\otimes -1} \tag{22}$$

is the α-fractional cosecant function.

$$tan_\alpha(x^\alpha) = sin_\alpha(x^\alpha) \otimes sec_\alpha(x^\alpha) \tag{23}$$

is the α-fractional tangent function. And

$$cot_\alpha(x^\alpha) = cos_\alpha(x^\alpha) \otimes csc_\alpha(x^\alpha) \tag{24}$$

is the α-fractional cotangent function.

Next, we introduce inverse fractional trigonometric functions.

Definition 2.12([17])**:** Suppose that $0 < \alpha \leq 1$. Then $arcsin_\alpha(x^\alpha)$ is the inverse function of $sin_\alpha(x^\alpha)$, and it is called inverse α-fractional sine function. $arccos_\alpha(x^\alpha)$ is the inverse function of $cos_\alpha(x^\alpha)$, and we say that it is the inverse α-

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fractional cosine function. On the other hand, $arctan_\alpha(x^\alpha)$ is the inverse function of $tan_\alpha(x^\alpha)$, and it is called the inverse α-fractional tangent function. $arccotan_\alpha(x^\alpha)$ is the inverse function of $cot_\alpha(x^\alpha)$, and it is the inverse α-fractional cotangent function. $arcsec_\alpha(x^\alpha)$ is the inverse function of $sec_\alpha(x^\alpha)$, and it is the inverse α-fractional secant function. $arccsc_\alpha(x^\alpha)$ is the inverse function of $csc_\alpha(x^\alpha)$, and it is called the inverse α-fractional cosecant function.

The main methods used in this paper are introduced below.

Theorem 2.13 ([17])**:** *Suppose that* $0<\alpha\leq 1$ *and* $\left|\frac{1}{\Gamma(\alpha+1)}x^\alpha\right|<1$. *Then*

$$\left({}_0D_x^\alpha\right)[arcsin_\alpha(x^\alpha)]=\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right]^{\otimes\left(-\frac{1}{2}\right)}, \tag{25}$$

$$\left({}_0D_x^\alpha\right)[arccos_\alpha(x^\alpha)]=-\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right]^{\otimes\left(-\frac{1}{2}\right)}, \tag{26}$$

$$\left({}_0D_x^\alpha\right)[arctan_\alpha(x^\alpha)]=\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right]^{\otimes(-1)}, \tag{27}$$

$$\left({}_0D_x^\alpha\right)[arccot_\alpha(x^\alpha)]=-\left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right]^{\otimes(-1)}. \tag{28}$$

Theorem 2.14 ([17])**:** *Let* $0<\alpha\leq 1$, *then*

$$arcsin_\alpha(x^\alpha)+arccos_\alpha(x^\alpha)=\frac{T_\alpha}{4}, \tag{29}$$

and

$$arctan_\alpha(x^\alpha)+arccot_\alpha(x^\alpha)=\frac{T_\alpha}{4}. \tag{30}$$

Theorem 2.15 ([14])**:** *If* $0<\alpha\leq 1$ *and* r *is a real number, then the* α*-fractional binomial series*

$$\left(1+\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes r}=\sum_{k=0}^{\infty}\frac{(r)_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}=\sum_{k=0}^{\infty}\frac{(r)_k}{\Gamma(k\alpha+1)}x^{k\alpha}, \tag{31}$$

and

$$\left(1-\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes r}=\sum_{k=0}^{\infty}\frac{(-1)^k(r)_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}=\sum_{k=0}^{\infty}\frac{(-1)^k(r)_k}{\Gamma(k\alpha+1)}x^{k\alpha}. \tag{32}$$

Where $(r)_k=r(r-1)\cdots(r-k+1)$ *for any positive integer* k, $(r)_0=1$, *and* $\left|\frac{1}{\Gamma(\alpha+1)}x^\alpha\right|<1$.

III. MAIN RESULTS

Theorem 3.1: *If* $0<\alpha\leq 1$, x *is a real number, and* $\left|\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}\right|<1$. *Then the* α*-fractional Taylor series of some inverse* α*-fractional trigonometric functions*

$$arcsin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{[(2k)!]^2}{2^{2k}(k!)^2\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}, \tag{33}$$

$$arccos_\alpha(x^\alpha)=\frac{T_\alpha}{4}-\sum_{k=0}^{\infty}\frac{[(2k)!]^2}{2^{2k}(k!)^2\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}, \tag{34}$$

$$arctan_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}, \tag{35}$$

$$arccot_\alpha(x^\alpha)=\frac{T_\alpha}{4}-\sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}. \tag{36}$$

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Proof: $arcsin_\alpha(x^\alpha) = \left({}_0I_x^\alpha\right)\left[\left[1-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right]^{\otimes\left(-\frac{1}{2}\right)}\right]$ (by Theorem 2.13)

$$= \left({}_0I_x^\alpha\right)\left[\sum_{k=0}^{\infty}\frac{(-1)^k\left(-\frac{1}{2}\right)_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}\right] \text{ (by Theorem 2.15)}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\left(-\frac{1}{2}\right)_k}{k!}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}\right]$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\left(-\frac{1}{2}\right)_k}{k!(2k+1)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(2k+1)}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\left(-\frac{1}{2}\right)_k(2k+1)!}{k!(2k+1)\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}$$

$$= \sum_{k=0}^{\infty}\frac{[(2k)!]^2}{2^{2k}(k!)^2\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha} \;.$$

Moreover, since $arcsin_\alpha(x^\alpha) + arccos_\alpha(x^\alpha) = \frac{T_\alpha}{4}$, it follows that

$$arccos_\alpha(x^\alpha) = \frac{T_\alpha}{4} - \sum_{k=0}^{\infty}\frac{[(2k)!]^2}{2^{2k}(k!)^2\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}.$$

On the other hand,

$$\left({}_0D_x^\alpha\right)[arctan_\alpha(x^\alpha)] = \left[1+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right]^{\otimes(-1)} \text{ (by Theorem 2.13)}$$

$$= \left({}_0I_x^\alpha\right)\left[\sum_{k=0}^{\infty}\frac{(-1)_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}\right] \text{ (by Theorem 2.15)}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k k!}{k!}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2k}\right]$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(2k+1)}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha} \;.$$

Furthermore, since $(x^\alpha) + arccot_\alpha(x^\alpha) = \frac{T_\alpha}{4}$, it follows that

$$arccot_\alpha(x^\alpha) = \frac{T_\alpha}{4} - \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha} \;.$$

Q.e.d.

Notation 3.2: If s is a real number, then the smallest integer greater than or equal to s is denoted as $\lceil s\rceil$.

Theorem 3.3: *Assume that* $0<\alpha\le 1$, n *is a positive integer*, x *is a real number, and* $\left|\frac{2}{\Gamma(2\alpha+1)}x^{2\alpha}\right|<1$. *Then the* n*-th order fractional derivative of inverse* α*-fractional trigonometric functions are*

$$\left({}_0D_x^\alpha\right)^n[arcsin_\alpha(x^\alpha)] = \sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}\frac{[(2k)!]^2}{2^{2k}(k!)^2\Gamma((2k-n+1)\alpha+1)}x^{(2k-n+1)\alpha}\;, \tag{37}$$

$$\left({}_0D_x^\alpha\right)^n[arccos_\alpha(x^\alpha)] = -\sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}\frac{[(2k)!]^2}{2^{2k}(k!)^2\Gamma((2k-n+1)\alpha+1)}x^{(2k-n+1)\alpha}\;, \tag{38}$$

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$$\left({}_{0}D_{x}^{\alpha}\right)^{n}[arctan_{\alpha}(x^{\alpha})]=\sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}\frac{(-1)^{k}(2k)!}{\Gamma((2k-n+1)\alpha+1)}x^{(2k-n+1)\alpha}\,, \tag{39}$$

$$\left({}_{0}D_{x}^{\alpha}\right)^{n}[arccot_{\alpha}(x^{\alpha})]=-\sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}\frac{(-1)^{k}(2k)!}{\Gamma((2k-n+1)\alpha+1)}x^{(2k-n+1)\alpha}\,. \tag{40}$$

Proof $\left({}_{0}D_{x}^{\alpha}\right)^{n}[arcsin_{\alpha}(x^{\alpha})]=\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[\sum_{k=0}^{\infty}\frac{[(2k)!]^{2}}{2^{2k}(k!)^{2}\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}\right]$ (by Theorem 3.1)

$$=\sum_{k=0}^{\infty}\frac{[(2k)!]^{2}}{2^{2k}(k!)^{2}\Gamma((2k+1)\alpha+1)}\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[\frac{1}{\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}\right]$$

$$=\sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}\frac{[(2k)!]^{2}}{2^{2k}(k!)^{2}\Gamma((2k-n+1)\alpha+1)}x^{(2k-n+1)\alpha}\,.$$

$$\left({}_{0}D_{x}^{\alpha}\right)^{n}[arccos_{\alpha}(x^{\alpha})]=\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[\frac{T_{\alpha}}{4}-arcsin_{\alpha}(x^{\alpha})\right]$$

$$=-\sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}\frac{[(2k)!]^{2}}{2^{2k}(k!)^{2}\Gamma((2k-n+1)\alpha+1)}x^{(2k-n+1)\alpha}\,.$$

$$\left({}_{0}D_{x}^{\alpha}\right)^{n}[arctan_{\alpha}(x^{\alpha})]=\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[\sum_{k=0}^{\infty}\frac{(-1)^{k}(2k)!}{\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}\right]$$

$$=\sum_{k=0}^{\infty}(-1)^{k}(2k)!\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[\frac{1}{\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}\right]$$

$$=\sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}\frac{(-1)^{k}(2k)!}{\Gamma((2k-n+1)\alpha+1)}x^{(2k-n+1)\alpha}.$$

$$\left({}_{0}D_{x}^{\alpha}\right)^{n}[arccot_{\alpha}(x^{\alpha})]=\left({}_{0}D_{x}^{\alpha}\right)^{n}\left[\frac{T_{\alpha}}{4}-arctan_{\alpha}(x^{\alpha})\right]$$

$$=-\sum_{k=\left\lceil\frac{n-1}{2}\right\rceil}^{\infty}\frac{(-1)^{k}(2k)!}{\Gamma((2k-n+1)\alpha+1)}x^{(2k-n+1)\alpha}\,.$$

Q.e.d.

IV. CONCLUSION

As mentioned above, this paper obtained any order fractional derivative of inverse fractional trigonometric function based on Jumarie type of fractional derivative. The main methods we used are the fractional binomial series and a new multiplication of fractional analytic functions. In fact, these results in this study are generalizations of those in classical calculus. Moreover, the new multiplication is a natural operation of fractional analytic functions. In the future, we will continue to study the problems in applied mathematics and fractional calculus by using the new multiplication and the fractional binomial series.

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